EXAMPLES OF NON-ISOLATED BLOW-UP FOR PERTURBATIONS OF THE SCALAR CURVATURE EQUATION ON NON LOCALLY CONFORMALLY FLAT MANIFOLDS

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1. Introduction and statement of the results

Let (M,g) be a smooth compact Riemannian manifold of dimension $n\geq 3$. Given a sequence $(h_\varepsilon)_{\varepsilon>0}\in C^\infty(M)$, we are interested in the existence of multi peaks positive solutions $(u_\varepsilon)_{\varepsilon>0}\in C^\infty(M)$ to the family of critical equations

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star-1} \text{ in } M \text{ for all } \varepsilon>0, \tag{1}$$

where $\Delta_g := -\mathrm{div}_g(\nabla)$ is the Laplace-Beltrami operator, and $2^\star := \frac{2n}{n-2}$ is the critical Sobolev exponent. We say that the family $(u_\varepsilon)_\varepsilon$ blows up as $\varepsilon\to 0$ if $\lim_{\varepsilon\to 0}\|u_\varepsilon\|_\infty = +\infty$. Blowing-up families to equations like (1) are described precisely by Struwe [18] in the energy space $H_1^2(\mathrm{M})$: namely, if the Dirichlet energy of u_ε is uniformly bounded with respect to ε, then there exists $u_0\in C^\infty(M)$, there exists $k\in\mathbb{N}$, there exists k families $(\xi_{i,\varepsilon})_\varepsilon\in M$ and $(\mu_{i,\varepsilon})_\varepsilon\in(0,+\infty)$ such that

$$u_\varepsilon = u_0 + \sum_{i=1}^k \left(\frac{\sqrt{n(n-2)}\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2 + d_g(\cdot,\xi_{i,\varepsilon})^2}\right)^{\frac{n-2}{2}} + o(1), \tag{2}$$

where $\lim_{\varepsilon\to 0} o(1) = 0$ in $H_1^2(M)$ and $\lim_{\varepsilon\to 0}\mu_{i,\varepsilon} = 0$ for all $i=1,...,k$. In this situation, we say that u_ε develops k peaks when $\varepsilon\to 0$.

We say that $\xi_0\in M$ is a blow-up point for $(u_\varepsilon)_\varepsilon$ if $\lim_{\varepsilon\to 0}\max_{B_r(\xi_0)} u_\varepsilon = +\infty$ for all $r>0$. It follows from elliptic theory that the blow-up points of a family of solutions $(u_\varepsilon)_\varepsilon$ to (1) satisfying (2) is exactly $\{\lim_{\varepsilon\to 0}\xi_{i,\varepsilon} / \, i=1,..,k\}$.

Following the terminology introduced by Schoen [16], $\xi_0\in M$ is an isolated point of blow-up for $(u_\varepsilon)_\varepsilon$ if there exists $(\xi_\varepsilon)_\varepsilon\in M$ such that

- ξ_ε is a local maximum point of u_ε for all $\varepsilon>0$,
- $\lim_{\varepsilon\to 0}\xi_\varepsilon = \xi_0$,
- there exists $C,\bar r>0$ such that $d_g(x,\xi_\varepsilon)^{\frac{n-2}{2}}u_\varepsilon(x)\leq C$ for all $x\in B_{\bar r}(\xi_0)$,
- $\lim_{\varepsilon\to 0}\max_{B_r(\xi_0)} u_\varepsilon = +\infty$ for all $r>0$.

The notion has proved to be very useful in the analysis of critical equations. Let $c_n := \frac{n-2}{4(n-1)}$ and R_g be the scalar curvature of (M,g). Compactness for the Yamabe equation

$$\Delta_g u + c_n R_g u = u^{2^\star-1} \tag{3}$$

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when $n \leq 24$ (the full result is due to Kuhri–Marques–Schoen [10]) is established by proving first that the sole possible blow-up points for (3) are isolated, see Schoen [16, 17], Li–Zhu [13], Druet [5], Marques [14], Li–Zhang (Theorem 1.1 in [12]), and Kuhri–Marques–Schoen [10]). When $n \geq 25$, there are examples of non-compactness of equation (3) (Brendle [1] and Brendle-Marques [2]).

In this note, we address the questions to know whether or not blow-up solutions for (1) do exist, and whether or not they necessarily have isolated blow-up points. When $h_\varepsilon \leq c_n R_g$, blow-up does not occur for $n \leq 5$ as shown by Druet [5] (except for the conformal class of the round sphere). When the potential is allowed to be above the scalar curvature, blow-up is possible: we refer to Druet-Hebey [6] for examples of non-isolated blow-up on the sphere with C^1-perturbations of the scalar curvature term in (3), and to Esposito-Pistoia-Vétois [9] for examples of isolated blow-up on general compact manifolds with arbitrary smooth perturbations of the scalar curvature. We present in this note examples of non-isolated blow-up points for smooth perturbations of the scalar curvature term in (3). This is the subject of the following theorem.

Theorem 1.1. *Let $\mathbb{S}^p \times \mathbb{S}^q$, $p, q \geq 3$ be endowed with the standard product metric g. For any $\xi_0 \in \mathbb{S}^p \times \mathbb{S}^q$, and $r \in \mathbb{N}$, there exists $(h_\varepsilon)_{\varepsilon>0} \in C^\infty(\mathbb{S}^p \times \mathbb{S}^q)$ such that $\lim_{\varepsilon\to 0} h_\varepsilon = c_n R_g$ in $C^r(\mathbb{S}^p \times \mathbb{S}^q)$, and there exists $(u_\varepsilon)_{\varepsilon>0} \in C^\infty(\mathbb{S}^p \times \mathbb{S}^q)$ a family of positive solutions to*

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star-1} \text{ in } \mathbb{S}^p \times \mathbb{S}^q \text{ for all } \varepsilon > 0,$$

such that the u_ε's blow up at ξ_0 as $\varepsilon \to 0$ with an arbitrarily large number of peaks. In particular, ξ_0 is not an isolated blow-up point for the u_ε's.

As a consequence, when dealing with general perturbed equations like (1), one has to deal with the delicate situation of the accumulation of peaks at a single point. The C^0-theory by Druet-Hebey-Robert [8] addresses this question in the a priori setting and L^∞-norm. We refer also to Druet [4] and Druet and Hebey [7] where the analysis of the radii of interaction of multi peaks solutions is performed.

The choice of this note is to perturb the potential $c_n R_g$ of the equation. Another point of view is to fix the potential $c_n R_g$ and to multiply the nonlinearity $u^{2^\star-1}$ by smooth functions then leading to consider Kazdan-Warner type equations: in this slightly different context, Chen-Lin [3] and Brendle (private communication) have constructed non-isolated local blow-up respectively in the flat case and in the Riemannian case.

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2. Proofs

We prove the following theorem that covers Theorem 1.1:

Theorem 2.1. *Let (M, g) be a non-locally-conformally flat compact Riemannian manifold of dimension $n \geq 6$ with positive Yamabe invariant. We fix $\xi_0 \in M$ such that the Weyl tensor at ξ_0 is such that $Weyl_g(\xi_0) \neq 0$. We let $k \geq 1$ and $r \geq 0$ be*

two integers. Then there exists $(h_\varepsilon)_{\varepsilon>0} \in C^\infty(M)$ such that $\lim_{\varepsilon\to 0} h_\varepsilon = c_n R_g$ in $C^r(M)$, and there exists $(u_\varepsilon)_{\varepsilon>0} \in C^\infty(M)$ a family of solutions to

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star-1} \text{ in } M \text{ for all } \varepsilon > 0,$$

such that $(u_\varepsilon)_\varepsilon$ develops k peaks at the blow-up point ξ_0. Moreover, ξ_0 is an isolated blow-up point if and only if $k = 1$.

Note that the Weyl tensor of $\mathbb{S}^p \times \mathbb{S}^q$ endowed with the product metric never vanishes. The proof of Theorem 2.1 relies on a Lyapunov-Schmidt reduction. We fix $\xi_0 \in M$ such that $\mathrm{Weyl}_g(\xi_0) \neq 0$. It follows from the classical conformal normal coordinates theorem of Lee-Parker [11] that there exists $\Lambda \in C^\infty(M \times M)$ such that for any $\xi \in M$,

$$R_{g_\xi}(\xi) = 0,\ \nabla R_{g_\xi}(\xi) = 0, \text{ and } \Delta_{g_\xi} R_{g_\xi}(\xi) = \frac{1}{6}\mathrm{Weyl}_g(\xi),$$

where $\Lambda_\xi := \Lambda(\xi, \cdot)$ and $g_\xi := \Lambda_\xi^{4/(n-2)} g$. Without loss of generality, up to a conformal change of metric, we assume that $g_{\xi_0} = g$. We let $r_0 > 0$ be such that $r_0 < i_{g_\xi}(M)$ for all $\xi \in M$ compact, where $i_{g_\xi}(M)$ is the injectivity radius of M with respect to the metric g_ξ. We let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(x) = 1$ for $x \le r_0/2$ and $\chi(x) = 0$ for $x \ge r_0$. We define a bubble centered at ξ with parameter δ as:

$$W_{\delta,\xi} := \chi(d_g(\cdot,\xi))\Lambda_\xi \left(\frac{\sqrt{n(n-2)}\delta}{\delta^2 + d_{g_\xi}(\cdot,\xi)^2} \right)^{\frac{n-2}{2}}.$$

We fix an integer $k \ge 1$. Given $\alpha > 1$ and $K > 0$, we define the set

$$\mathcal{D}^{(k)}_{\alpha,K}(\delta) := \left\{ ((\delta_i)_i, (\xi_i)_i) \in (0,\delta)^k \times M^k \ \Big/ \ \frac{1}{\alpha} < \frac{\delta_i}{\delta_j} < \alpha \ ; \ \frac{d_g(\xi_i,\xi_j)^2}{\delta_i\delta_j} > K \text{ for } i \neq j \right\}.$$

For any $h \in C^0(M)$, we define the functional:

$$J_h(u) := \frac{1}{2}\int_M (|\nabla u|_g^2 + hu^2)\, dv_g - \frac{1}{2^\star}\int_M u_+^{2^\star}\, dv_g$$

for all $u \in H_1^2(M)$. For $((\delta_i)_i, (\xi_i)_i) \in \mathcal{D}^{(k)}_{\alpha,K}$, we define the error

$$R_{(\delta_i)_i,(\xi_i)} := \left\| (\Delta_g + h)\left(\textstyle\sum_{i=1}^k W_{\delta_i,\xi_i} \right) - \left(\textstyle\sum_{i=1}^k W_{\delta_i,\xi_i} \right)^{2^\star-1} \right\|_{\frac{2n}{n+2}}$$

The classical Lyapunov-Schmidt finite-dimensional reduction yields the following:

Proposition 2.2. *We fix $\alpha > 1$, $\eta > 0$ and $C_0 > 0$ such that $\|h\|_\infty \le C_0$ and $\lambda_1(\Delta_g + h) \ge C_0^{-1}$. Then there exists $K_0 = K_0((M,g),\alpha,C_0,\eta) > 0$, $\delta_0 = \delta_0((M,g),\alpha,C_0,\eta) > 0$ and $\phi \in C^1(\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0), H_1^2(M))$ such that*

- *$R_{(\delta_i)_i,(\xi_i)} < \eta$ for all $(\delta_i)_i, (\xi_i) \in \mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$,*
- *$u(h,(\delta_i)_i,(\xi_i)) := \sum_{i=1}^k W_{\delta_i,\xi_i} + \phi((\delta_i)_i,(\xi_i))$ is a critical point of J_h iff $((\delta_i)_i,(\xi_i))$ is a critical point of $((\delta_i)_i,(\xi_i)) \mapsto J_h(u((\delta_i)_i,(\xi_i)))$ in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$,*
- *$J_h(u(h,(\delta_i)_i,(\xi_i)_i)) = J_h(\sum_{i=1}^k W_{\delta_i,\xi_i}) + O(R^2_{(\delta_i)_i,(\xi_i)})$.*

Here, $|O(1)| \le C((M,g),\alpha,C_0)$ uniformly in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$.

This result is essentially contained in the existing litterature. We refer to Esposito-Pistoia-Vétois [9] and Robert-Vétois [15] for details concerning the proof of this proposition.

From now on, we fix $((\delta_i)_i, (\xi_i)_i) \in \mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$. Standard computations yield

$$J_h\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big) = \sum_{i=1}^k J_h(W_{\delta_i,\xi_i}) + \Big(\sum_{i\neq j}\int_M (\nabla W_{\delta_i,\xi_i}, \nabla W_{\delta_j,\xi_j})_g$$
$$+ hW_{\delta_i,\xi_i}W_{\delta_j,\xi_j}\,dv_g\Big) - \frac{1}{2^\star}\int_M \Big(\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big)^{2^\star} - \sum_{i=1}^k W^{2^\star}_{\delta_i,\xi_i}\Big)\,dv_g$$

and

$$\int_M \Big(\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big)^{2^\star} - \sum_{i=1}^k W^{2^\star}_{\delta_i,\xi_i}\Big)\,dv_g = O\Big(\sum_{i\neq j}\int_{W_{\delta_i,\xi_i}\leq W_{\delta_j,\xi_j}} W_{\delta_i,\xi_i}W^{2^\star-1}_{\delta_j,\xi_j}\,dv_g\Big).$$

Taking K_0 larger if necessary, careful estimates yield

$$J_h\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big) = \sum_{i=1}^k J_h(W_{\delta_i,\xi_i}) + O\Big(\sum_{i\neq j}\Big(\frac{\delta_i\delta_j}{d_g(\xi_i,\xi_j)^2}\Big)^{\frac{n-2}{2}}\Big)$$

and

$$R_{(\delta_i)_i,(\xi_i)} \leq \sum_{i=1}^k \|(\Delta_g+h)W_{\delta_i,\xi_i} - W^{2^\star-1}_{\delta_i,\xi_i}\|_{\frac{2n}{n+2}} + O\Big(\sum_{i\neq j}\Big(\frac{\delta_i\delta_j}{d_g(\xi_i,\xi_j)^2}\Big)^{\frac{n-2}{4}}\Big)$$

uniformly in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$. Moreover, see Proposition 2.3 in Esposito-Pistoia-Vétois [9], we have that

$$J_h(W_{\delta,\xi}) = \frac{K_n^{-n}}{n}\Big(1 + \frac{2(n-1)}{(n-2)(n-4)}(h - c_nR_g)(\xi)\delta^2 + O(\|h-c_nR_g\|_{C^1})\delta^3$$
$$- |Weyl_g(\xi)|_g^2 \begin{cases} \frac{1}{64}\delta^4\ln\frac{1}{\delta} + O(\delta^4) & \text{when } n=6 \\ \frac{1}{24(n-4)(n-6)}\delta^4 + O(\delta^5) & \text{when } n\geq 7 \end{cases}\Big)$$

and

$$\|(\Delta_g+h)W_{\delta,\xi} - W^{2^\star-1}_{\delta,\xi}\|_{\frac{2n}{n+2}} \leq C\delta^2 \begin{cases} 1 + \|h-c_nR_g\|_{C^0}\left(\ln\frac{1}{\delta}\right)^{2/3} & \text{when } n=6 \\ \sqrt{\delta} + \|h-c_nR_g\|_{C^0} & \text{when } n\geq 7. \end{cases}$$

Here again, $|O(1)| \leq C((M,g),\alpha,C_0)$ uniformly in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$.

We now choose the $(\delta_i),(\xi_i)'s$ and the function h. For any $\varepsilon>0$, we let $\delta_\varepsilon>0$ be such that

$$\delta_\varepsilon^2\ln\frac{1}{\delta_\varepsilon} = \varepsilon \text{ when } n=6 \text{ and } \delta_\varepsilon^2 = \varepsilon \text{ when } n\geq 7.$$

We let $H\in C^\infty(\mathbb{R}^n)$ be such that

- $H(x) = -1$ for all $|x|>2$,
- H admits k distinct strict local maxima at $p_{i,0}\in B_1(0)$ for $i=1,...,k$,
- $H(p_{i,0})>0$ for all $i=1,...,k$.

We let $\tilde r>0$ be such that for any $i\in\{1,...,k\}$, the maximum of H on $B_{2\tilde r}(p_{i,0})$ is achieved exactly at $p_{i,0}$ and such that $|p_{i,0}-p_{j,0}|\geq 3\tilde r$ for all $i\neq j$. We let $(\mu_\varepsilon)_\varepsilon\in(0,+\infty)$ be such that $\lim_{\varepsilon\to 0}\mu_\varepsilon = 0$ and

$$(|\ln\varepsilon|)^{-1/4} = o(\mu_\varepsilon) \text{ when } n=6 \text{ and } \varepsilon^{\frac{n-6}{2(n-2)}} = o(\mu_\varepsilon) \text{ when } n\geq 7,$$

where both limits are taken when $\varepsilon \to 0$. We define

$$h_\varepsilon(x) := c_n R_g(x) + \varepsilon H\left(\mu_\varepsilon^{-1}\exp_{x_0}^{-1}(x)\right) \text{ for all } x \in M.$$

Here, the exponential map is taken with respect to the metric g and after assimilation to $\mathbb{R}^n$ of the tangent space at ξ_0: this definition makes sense for $\varepsilon > 0$ small enough. For $(t_i)_i \in (0,+\infty)^k$ and $(p_i)_i \in (\mathbb{R}^n)^k$, we define

$$\tilde{u}_\varepsilon((t_i)_i,(p_i)_i) := u\left(h_\varepsilon, (t_i\delta_\varepsilon)_i, (\exp_{\xi_0}(\mu_\varepsilon p_i))_i\right).$$

The above estimates and the choice of the parameters yields
(4)

$$\lim_{\varepsilon\to 0}\frac{J_{h_\varepsilon}(\tilde{u}_\varepsilon((t_i)_i,(p_i)_i)) - k\frac{K_n^{-n}}{n}}{\varepsilon\delta_\varepsilon^2} = \sum_{i=1}^k F_n(t_i,p_i) \text{ in } C^0_{loc}((0,+\infty)^k \times \prod_{i=1}^k B_r(p_{i,0}))$$

where

$$F_n(t,p) := \frac{2(n-1)}{(n-2)(n-4)}H(p)t^2 - d_n|Weyl_g(\xi_0)|_g^2 t^4 \text{ for } (t,p) \in (0,+\infty)\times\mathbb{R}^n$$

with $d_6 = \frac{1}{64}$ and $d_n := \frac{1}{24(n-4)(n-6)}$ for $n \geq 7$. As easily checked, up to choosing the $t_i's$ in suitable compact intervalles $I_1,...,I_k$, the right-hand-side of (4) has a unique maximum point in the interior of $\prod_{i=1}^k I_i \times \prod_{i=1}^k B_{\tilde{r}}(p_{i,0})$. As a consequence, for $\varepsilon > 0$ small enough, $J_{h_\varepsilon}(\tilde{u}_\varepsilon((t_i)_i,(p_i)_i))$ admits a critical point, $((t_{i,\varepsilon})_i,(p_{i,\varepsilon})_i) \in (\alpha,\beta)^k \times \prod_{i=1}^k B_{\tilde{r}}(p_{i,0})$ for some $0 < \alpha < \beta$ independent of ε. Defining $u_\varepsilon := \tilde{u}_\varepsilon((t_{i,\varepsilon})_i,(p_{i,\varepsilon})_i)$, it follows from Proposition 2.2 and the strong maximum principle that

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star-1} \text{ in } M$$

for $\varepsilon > 0$ small enough. In addition to the hypotheses above, we require that $\varepsilon = o(\mu_\varepsilon^r)$ when $\varepsilon \to 0$, which yields $\lim_{\varepsilon\to 0} h_\varepsilon = c_n R_g$ in $C^r(M)$.

We prove that $(u_\varepsilon)_\varepsilon$ develops no isolated blow-up point when $k \geq 2$. We argue by contradiction. Moser's iterative scheme yields the convergence of u_ε to 0 in $C^2_{loc}(M\setminus\{\xi_0\})$. We then get that the isolated blow-up point is ξ_0, and thus that there exists $r_1 > 0$ and $(\xi_\varepsilon)_\varepsilon \in M$ such that $\lim_{\varepsilon\to 0}\xi_\varepsilon = \xi_0$ and there exists $C > 0$ such that

$$d_g(x,\xi_\varepsilon)^{\frac{n-2}{2}}u_\varepsilon(x) \leq C \text{ for all } \varepsilon > 0 \text{ and } x \in B_{r_1}(\xi_0). \tag{5}$$

For any $i = 1,..,k$, we define $\xi_{i,\varepsilon} := \exp_{\xi_0}(\mu_\varepsilon p_{i,\varepsilon})$ and

$$\tilde{u}_{i,\varepsilon}(x) := (\delta_\varepsilon t_{i,\varepsilon})^{\frac{n-2}{2}}u_\varepsilon(\exp_{\xi_{i,\varepsilon}}(\delta_\varepsilon t_{i,\varepsilon}x))$$

for all $|x| < r_0/(2\delta_\varepsilon t_{i,\varepsilon})$. It follows from standard elliptic theory that

$$\lim_{\varepsilon\to 0}\tilde{u}_{i,\varepsilon} = \left(\frac{\sqrt{n(n-2)}}{1+|\cdot|^2}\right)^{\frac{n-2}{2}} \text{ in } C^2_{loc}(\mathbb{R}^n). \tag{6}$$

Moreover, if $\delta_\varepsilon = o(d_g(\xi_{i,\varepsilon},\xi_\varepsilon))$ when $\varepsilon \to 0$, inequality (5) yields the convergence of $\tilde{u}_{i,\varepsilon}$ in $C^0_{loc}(\mathbb{R}^n)$: a contradiction with (6). Therefore, $d_g(\xi_\varepsilon,\xi_{i,\varepsilon}) = O(\delta_\varepsilon)$ when $\varepsilon \to 0$ for all $i = 1,...,k$, and then $d_g(\xi_{i,\varepsilon},\xi_{j,\varepsilon}) = O(\delta_\varepsilon) = o(\mu_\varepsilon)$ when $\varepsilon \to 0$ for all $i \neq j$. This contradicts the fact that $d_g(\xi_{i,\varepsilon},\xi_{j,\varepsilon}) \geq c_0\mu_\varepsilon$ when $k \geq 2$. This proves the non-simple blow-up when $k \geq 2$.

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